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ORIGINAL PAPER

The wavelet methods to linear and nonlinear reaction–diffusion model arising in mathematical chemistry

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Abstract In this paper, we have applied an accurate and efficient wavelet scheme (due to Legendre polynomial) to find the numerical solutions for a set of coupled reaction–diffusion equations. This technique provides the solutions in rapid convergence series with computable terms for the problems with high degree of non linear terms appearing in the governing differential equations. The highest derivative in the differential equation is expanded into wavelet series, this approximation is then integrated while the boundary conditions are applied by using integration constants. With the help of operational matrices, the nonlinear reaction–diffusion equations are converted into a system of algebraic equations. Finally, some numerical examples to demonstrate the validity and applicability of the method have been furnished. The use of Legendre wavelets is found to be accurate, efficient, simple, and computationally attractive. This wavelet method can be used for obtaining quick solution in many chemical Engineering problems.

Keywords Nonlinear reaction–diffusion equations · Operational matrices · Haar wavelets · Legendre wavelets · Laplace transform method · Homotopy analysis method

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1 Introduction

In recent years, nonlinear reaction–diffusion equations (NLRDEs) have been widely studied and applied in science and engineering. This study concerns the numerical solutions of nonlinear reaction–diffusion modelling the dynamics of diffusion and nonlinear reproduction for a population. The associated NLRDEs was initiated by Fisher [1] to describe the propagation behaviour of a virile mutant. Kolmogoroff et al. [2] introduced the problem admits a wave solution with minimum speed. The NLRDEs describe a population of diploid individuals. These equations have wide application in the fields of logistic population growth, flame propagation, euro physiology, autocatalytic chemical reactions, branching Brownian motion processes, and nuclear reactor theory. But these non linear PDEs are difficult to get their exact solutions. So the approximation and numerical methods must be used. The numerical solutions of the NLRDEs have received considerable attention in the literature and fall into two groups: The analytical methods and the numerical ones. Analytical methods enable researchers to study the effect of different variables or parameters on the function under study easily. Recently, there are many new algoritms for NLRDEs have been proposed, for example, the adomian decomposition method (ADM) [3], the variational iteration method [4], differential transform method, Reduced differential transform method [5], the Homotopy perturbation method [6], the homotopy analysis method (HAM) [7] and other methods. Wang [8] applied the tanh method for Fisher's equation, which represents shock waves structures. Mansour [9] showed that traveling wave solutions of a nonlinear reaction–diffusion-chemotaxis model for bacterial pattern formation. Daniel Olmos and Bernie Shizgal [10] established the Pseudo-spectral method of solution of Fisher's equation. Wazwaz [11] introduced the analytical study on Burgers, Fisher, and Huxley equations and combined forms of these equations. Recently, Alam Khan et al. [12] introduced the approximate analytical solutions of fractional reaction–diffusion equations.

Rajendran and Senthamarai [13] solved the nonlinear coupled reaction–diffusion equation by the He's variational iteration method. Baronas et al. [14] had developed the mathematical modeling of flow injection analysis and they solved the model equations by the finite difference method. In recent years, non linear reaction diffusion equations (NLRDE) have been used as a basis for a wide variety of models, for the special spread of gene in population and for chemical wave propagation. Mavoungou and Cherault [15] showed ADM for solving Fishers equations. Wazwaz and Gorguis [16] applied the ADM for solving Fisher type equation. Malfliet [17] introduced the travelling wave solutions of complicated nonlinear PDEs. In recent years, there have been huge activities in developing the approximation and analytical methods for Fisher's equations. Explicit solutions of the Fisher's equation for a special wave speed obtained by Ablowitz and Zeppetella [18]. Puri et al. [19] applied the singular peturbation Method for Fisher's equation. Carey and Shen [20] implemented the least square Finite element method for Fisher's reaction diffusion equation. Al-Khaled [21] introduced the sinc-collocation method by the Pseudo-spectral method for the numerical solution of Fisher's equation. Recently, Mittal

and Jiwari [22] have presented the differential quadrature method for Fisher's equations.

Wavelets theory is a relatively new and as emerging area in applied mathematical research. It has been applied many different field of science and Engineering. Moreover wavelet transform establishes a connection with efficient and fast numerical algorithms.

In recent years, wavelet transforms have found their way into many different fields in science, engineering and medicine. Wavelet analysis or wavelet theory, as a relatively new and an emerging area in applied mathematical research, has received considerable attention in dealing with NLRDEs. It possesses many useful properties, such as Compact support, orthogonality, dyadic, and orthonormality and multi-resolution analysis (MRA). Recently, Haar wavelets have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

In the numerical analysis, wavelet based methods and hybrid methods become important tools because of the properties of localization. In wavelet based methods, there are two important ways of improving the approximation of the solutions: Increasing the order of the wavelet family and the increasing the resolution level of the wavelet. There is a growing interest in using various wavelets [23] to study problems, of greater computational complexity. Among the wavelet transform families the Haar and Legendre wavelets deserve much attention. The basic idea of Legendre wavelet method is to convert the Partial differential equations to a system of algebraic equations by the operational matrices of integral or derivative [24]. The main goal is to show how wavelets and MRA can be applied for improving the method in terms of easy implementability and achieving the rapidity of its convergence. Recently, Hesameddini and Shekarpaz [25] used the Legendre wavelet method for solving Klein–Gordon equations. Razzaghi and Yousefi [26] introduced the Legendre wavelet method for solving variational problems and constrained optimal control problems. Hariharan et al. [27] had introduced the diffusion equation, convection-diffusion equation, Reaction–diffusion equation, nonlinear parabolic equations, fractional Klein–Gordon equations, Sine–Gordon equations and Fisher's equation by the Haar wavelet method. Liu and Lin En-bing [28] applied the Legendre wavelet method for solving partial differential equations. Mohammadi and Hosseini [29] had showed a new Legendre wavelet operational matrix of derivative in solving singular ordinary differential equations. Parsian [30] introduced two dimensional Legendre wavelets and operational matrices of integration. Ben-yu et al. [31] implemented a Legendre spectral method for solving non-linear Klien–Gordon equation. Mohamadi et al. [32] used the Legendre wavelets for fractional order boundary value problems. For complete sake of Legendre wavelet is presented in the literature [23].

Razzaghi and Yousefi [33] introduced the Legendre wavelets operational matrix of integration. Yousefi [34] applied the Legendre wavelets for solving differential

equations of Lane-Emden type. Maleknejad and Sohrabi [35] had showed the numerical solutions of Fredholm integral equations of first kind by the Legendre wavelets method. Hariharan et al. [36] had used the Haar wavelet method for estimating the depth profile of soil temperature. Hariharan and Kannan [1] established the Haar wavelet method for solving Cahn-Allen equation. Jafari et al. [37] used the Legendre wavelets method for solving fractional differential equations. Yin Yang [38] applied the Legendre pseudo spectral method for solving multi-order fractional differential equations. In the numerical analysis, wavelet based methods and hybrid methods become important tools because of the properties of localization. In wavelet based methods, there are two important ways of improving the approximation of the solutions: Increasing the order of the wavelet family and the increasing the resolution level of the wavelet. There is a growing interest in using wavelets [39–44] to study problems, of greater computational complexity. Among the wavelet transform families the Haar and Legendre wavelets deserve much attention. Lepik [39] applied the Hare wavelet method for evolution equations. In recent years, several analytical/approximation methods are implemented for solving Fisher-type PDEs (see Ref. [45–47]).

In this paper we have applied the wavelet methods for solving a fisher type model equations arising in mathematical chemistry.

This paper is organized as follows: In Sect. 2, the mathematical formulation of the problem is presented. The Haar and Legendre wavelets are demonstrated in Sect. 3. Then, the methods of solution for NLRDEs are implemented in Sect. 4. In Sect. 5, the convergence analysis is described. Several numerical examples to demonstrate the effectiveness of the proposed method are given in Sect. 6. Concluding remarks are given in Sect. 7.

2 Mathematical formulation of the problem

Consider the following the enzyme-catalyzed reaction $S \xrightarrow{E} P$ characterized by the substrate (S) binding to the enzyme (E) causes for the product (P) while the rate of product appearance depends on concentration substrate, under the following assumptions.

(i) Diffusion of substrate molecules is neglected
(ii) Enzyme reaction is at steady state, the mathematical model for enzyme kinetics is given by Michaelis–Menten equation

$$\nu = \frac{dP}{dt} = -\frac{dS}{dt} = \frac{V_{\max} S}{K_M + S} \tag{2.1}$$

where ν is the rate of the enzymatic reaction, $V_{\max}$ is the maximal enzymatic rate attainable with that amount of enzyme, when the enzyme is fully saturated with substrate, K_M is the Michaelis constant, S is the substrate concentration, P is concentration of the reaction product, and t is time. $V_{\max}$ corresponds to relative activity of substrate.
(iii) Symmetrical geometry of the electrode.

(iv) Homogeneous distribution of immobilized enzyme in the enzyme membrane. Using Fick's law, the non-linear reaction–diffusion equation with diffusion described by the following equations [13]

$$\frac{\partial S}{\partial T} = D_S \frac{\partial^2 S}{\partial X^2} - \frac{V_{\max} S}{K_M + S}, \quad 0 < X < 1, 0 < T \leq \tau \tag{2.2}$$

$$\frac{\partial P}{\partial T} = D_P \frac{\partial^2 P}{\partial X^2} + \frac{V_{\max} S}{K_M + S}, \quad 0 < X < 1, 0 < T \leq \tau, \tag{2.3}$$

where τ is full time of biosensor operation to be analyzed, D_S and D_P are diffusion coefficients of the substrate and product respectively.

The operation of biosensor starts when some substrate appears over the surface of the enzyme layer. Thus initial conditions take the form

$$S(x, 0) = 0, \quad 0 \leq x < 1$$
$$P(x, 0) = 0, \quad 0 \leq x \leq 1$$

$S(d, 0) = S_0$, where S_0 is the concentration of substrate (bulk) over the biosensor and d is thickness of enzyme layer. If the substrate is well stirred and in powerful motion, then the diffusion layer $(0 < X < d)$ will remain at a constant thickness. When the substrate (analyte) disappears, a buffer solution swills the enzyme surface, reducing the substrate concentration at this surface to zero. The substrate (analyte) remaining in the enzyme membrane, the mass diffusion as well as the reaction still continues some time even after the disconnect of the biosensor and substrate. Therefore, we have used the boundary conditions in the interval $0 < T \leq \tau$ given by

$$\left(\frac{\partial S}{\partial x}\right)_{x=0} = 0$$
$$S(d, T) = \begin{cases} S_0, & T \leq \tau_F \\ 0, & T > \tau_F \end{cases}$$
$$P(0, T) = P(d, T) = 0$$

where τ_F is the time of flow injection, i.e., the time when analyte disappears from the bulk solution/membrane interface.

By applying the following parameters,

$$s = \frac{S}{kS^\infty}, p = \frac{P}{kS^\infty}, x = \frac{X}{L},$$
$$t = \frac{D_s T}{L^2}, \alpha = \frac{kS^\infty}{K_M}, K = \frac{V_{\max} L^2}{K_M D_s}$$

We obtained the following dimensionless NLRDEs,

$$\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - \frac{Ks}{1 + \alpha s}, 0 < t < 1, 0 < t \leq \tau \tag{2.4}$$

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \frac{Ks}{1+\alpha s}, 0 < t < 1, 0 < t \leq \tau, \tag{2.5}$$

where s and p represents the dimensionless concentration of substrate and product while x and t represents dimensionless distance and time parameter, respectively. The parameter α denotes the saturation of the enzyme and K denotes reaction–diffusion parameter. When $\alpha s \ll 1$, Eqs. (4) and (5) are in the from [See Ref. [13])

$$\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - Ks(1-\alpha s) \tag{2.6}$$

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + Ks(1-\alpha s) \tag{2.7}$$

since $s = 1 - p$, it is enough to solve Eq. (2.6) only. This type of NLRDE comes across in chemical kinetics, population dynamics and non-linear waves. Such non-linear equations also occur often in the description of chemical and biological phenomena.

3 Haar and Legendre wavelets and properties

3.1 Wavelets

Wavelets are the family of functions which are derived from the family of scaling function $\{\emptyset_{j,k:}\, k \in Z\}$ where

$$\emptyset(x) = \sum_k a_k \quad \emptyset(2x-k) \tag{3.1}$$

For the continuous wavelets, the following equation can be represented:

$$\Psi_{a,b}(x) = |a|^{\frac{-1}{2}}\, \Psi\left(\frac{x-b}{a}\right)\ a, b \in R, a \neq 0. \tag{3.2}$$

where a and b are dilation and translation parameters, respectively, such that $\Psi(x)$ is a single wavelet function.

The discrete values are put for a and b in the initial form of the continuous wavelets, i.e.:

$$\mathrm{a} = \mathrm{a}_0^{-\mathrm{j}},\ \mathrm{a}_0 > 1,\ \mathrm{b}_0 > 1, \tag{3.3}$$

$$b = kb_0a_0^{-j}, \quad j,k \in Z. \tag{3.4}$$

Then, a family of discrete wavelets can be constructed as follows:

$$\Psi_{j,k} = |a_0|^{\frac{1}{2}}\, \Psi(2^j x - k), \tag{3.5}$$

So, $\Psi_{j,k}(x)$ constitutes an orthonormal basis in L^2 (R), where $\Psi(x)$ is a single function.

3.2 Haar wavelets preliminaries

Haar wavelet was a system of square wave; the first curve was marked up as $h_0(t)$, the second curve marked up as $h_1(t)$ that is

$$h_0(x) = \begin{cases} 1, \ 0 \le x < 1 \\ 0, \ otherwise \end{cases}$$

$$h_1(x) = \begin{cases} 1, & 0 \le x < 1/2, \\ -1, & 1/2 \le x < 1, \\ 0, & otherwise, \end{cases}$$

where $h_0(x)$ is scaling function, $h_1(x)$ is mother wavelet. In order to perform wavelet transform, Haar wavelet uses dilations and translations of function, i.e. the transform make the following function.

$$h_n(x) = h_1(2^j x - k), \quad n = 2^j + k, \ j \ge 0, \ 0 \le k < 2^j. \tag{3.6}$$

The first eight Haar function and their integrals are presented in Ref. [23]

3.3 Function approximation

Any square integrable function $y(x) \in L^2[0, 1)$ can be expanded by a Haar series of infinite terms

$$y(x) = \sum_{i=0}^{\infty} c_i h_i(x), \ \ i \in \{0\} \cup N, \tag{3.7}$$

where the Haar coefficients c_i are determined as,

$$c_0 = \int_0^1 y(x) h_0(x) dx, \quad c_n = 2^j \int_0^1 y(x) h_i(x) dx, \ \ i = 2^j + k, \ j \ge 0, \ 0 \le k < 2^j, \ x \in [0, 1)$$

such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 \left[y(x) - \sum_{i=0}^{m-1} c_i h_i(x) \right]^2 dx, \ \ m = 2^j, \ \ j \in \{0\} \cup N. \tag{3.8}$$

Usually, the series expansion contains infinite terms for smooth $y(x)$. If $y(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite m terms, that is

$$y(x) = \sum_{i=0}^{m-1} c_i h_i(x) = c_{(m)}^T h_{(m)}(x) \tag{3.9}$$

where the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as

$$c_{(m)}^T = [c_0, c_1, \ldots, c_{m-1}] \tag{3.10}$$

and $h_{(m)}(x) = [h_0(x), h_1(x), \ldots, h_{m-1}(x)]^T$ where 'T' means transpose and $m = 2^j$.

The first four Haar function vectors, which $x = n/8$, $n = 1, 3, 5, 7$ can be expressed as follows

$$h_4(1/8) = [1, 1, 1, 0]^T, \quad h_4(3/8) = [1, 1, -1, 0]^T,$$
$$h_4(5/8) = [1, -1, 0, 1]^T, \quad h_4(7/8) = [1, -1, 0, -1]^T,$$

which can be written in matrix form as

$$H_4 = [h_4(1/8), h_4(3/8), h_4(5/8), h_4(7/8)]$$
$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

In general, we have

$$H_m = [h_m(1/2m), h_m(3/2m), \ldots, h_m(2m-1)/2m],$$

where $H_1 = [1]$, $H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The collocation points are identified as $x_l = (2l-1)/2m, l = 1, 2, \ldots, m$. In application, in order to avoid dealing with impulse function, integration of the vector $h_m(x)$ given by

$$\int_0^x h_m(t)dt \approx P_m h_m(x), \ \ x \in [0, 1], \tag{3.11}$$

where P_m is the $m \times m$ operational matrix and is given by

$$P_{(m)} = \frac{1}{2m} \begin{pmatrix} 2m P_{(m/2)} & -H_{(m/2)} \\ H_{(m/2)}^{-1} & O \end{pmatrix} \tag{3.12}$$

where O is a null matrix of order $\frac{m}{2} \times \frac{m}{2}$.

(The proof can be found in [41]) where $P_1 = [1/2]$, so

$$P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad P_4 = \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix},$$

$$P_8 = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chen and Hsiao [41] introduced that the following matrix equation for calculating the matrix P of order m holds

$$P_{(m)} = \frac{1}{2m}\begin{pmatrix} 2m P_{(m/2)} & -H_{(m/2)} \\ H^{-1}_{(m/2)} & O \end{pmatrix}$$

where O is a null matrix of order $\frac{m}{2} \times \frac{m}{2}$,

$$H_{m\times m} \underline{\underline{\Delta}}[h_m(t_0)\ h_m(t_1) - - - - - - h_m(t_{m-1})] \tag{3.13}$$

Here $\frac{i}{m} \le t < i + \frac{1}{m}$ and

$$H^{-1}_{mxm} = \frac{1}{m} H^T_{mxm}\ diag(r) \tag{3.14}$$

3.4 Legendre wavelets and its properties

The Legendre wavelets are defined by

$$\Psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}} L_m\left(2^k t - n^{\hbar}\right), & \text{for } \frac{n^n-1}{2^k} \le t \le \frac{n^n+1}{2^k} \\ 0, & \text{otherwise} \end{cases}, \tag{3.15}$$

where m $= 0, 1, 2, \ldots, M-1$ and k $= 1, 2, \ldots, 2^{j-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, then, the wavelets $\Psi_{k,m}(x)$ form an orthonormal basis for $L^2[0,1]$ [26]. In the above formulation of Legendre wavelets, the Legendre polynomials are in the following way:

$$p_0 = 1,$$
$$p_1 = x,$$
$$p_{m+1}(x) = \frac{2m+1}{m+1} x\, p_m(x) - \frac{m}{m+1} p_{m-1}(x). \tag{3.16}$$

and $\{p_{m+1}(x)\}$ are the orthogonal functions of order m, which is named the well-known shifted Legendre polynomials on the interval [0,1]. Note that, in the general form of Legendre wavelets, the dilation parameter is $a = 2^{-j}$ and the translation parameter is $b = n\, 2^j$.

3.5 Two-dimensional Legendre wavelets

Two-dimensional Legendre wavelets in $L^2(R)$ over the interval [0,1] × [0,1] as the form [26]:

$$\Psi_{n,m,n',m'}(x, y) = \begin{cases} \sqrt{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)} 2^{\frac{k+k'}{2}} p_m(x) p_{m'}(y), \\ \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \frac{n'-1}{2^{k-1}} \le y \le \frac{n'}{2^{k'-1}}; \\ 0, \text{otherwise.} \end{cases} \tag{3.17}$$

and $m = 0, 1, 2, \ldots, M-1$, $m' = 0, 1, 2, 3, \ldots M'-1$, $n = 1, 2, \ldots, 2^{k-1}$, $n' = 1, 2, \ldots 2^{k'-1}$ where

$$P_m(x) = \overline{P_{m'}}(2^k x - 2n + 1),\ P_{m'}(y) = \overline{P_{m'}}(2^{k'} y - 2n' + 1), \tag{3.18}$$

$\overline{P_m}$ are Legendre functions of order m defined over the interval $[-1,1]$.

By using two-dimensional shifted Legendre polynomials into $x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ and $y \in \left[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\right]$, the $\int_0^1 \Psi_{n,m,n',m'}(x, y)$ can be written as

$$\int_0^1 \Psi_{n,m,n',m'}(x, y) = A_{m,m'} . P_{m'}(x) P_{m'}(y) \chi_{\left[\begin{smallmatrix} \frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \\ \frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}} \end{smallmatrix}\right]}(x, y), \tag{3.19}$$

In which $A_{m,m'} = \sqrt{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)} 2^{\frac{k+k'}{2}}$ and $\chi_{\left[\begin{smallmatrix} \frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \\ \frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}} \end{smallmatrix}\right]}(x, y)$ is a characteristic function defined as $\chi_{\left[\begin{smallmatrix} \frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \\ \frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}} \end{smallmatrix}\right]}(x, y) = \begin{cases} 1, x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right], \\ y \in \left[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\right]; \\ 0, \ otherwise \end{cases}$

Two dimension Legendre Wavelets are an orthonormal set over [0,1] × [0,1].

$$\int_0^1 \int_0^1 \Psi_{n,m,n',m'}(x, y) \Psi_{n_1,m_1,n'_1,m'_1}(x, y) dx dy = \delta_{n,n_1} \delta_{n',n'_1} \delta_{m',m'_1} \tag{3.20}$$

The function u(x,y) $\in L^2(R)$ defined over [0,1] × [0,1] may be expanded as

$$\mathrm{u(x,y)} = \mathrm{X(x)Y(y)} \cong \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty} c_{n,m,n',m'}\ \Psi_{n,m,n',m'}(x,y) \tag{3.21}$$

If the infinite series in Eq. (3.21) is truncated, then Eq. (3.21) can be written as

$$\mathrm{u(x,y)} = \mathrm{X(x)Y(y)} \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{M'-1} c_{n,m,n',m'}\ \Psi_{n,m,n',m'}(x,y) \tag{3.22}$$

where $c_{n,m,n',m'} = \int_0^1\int_0^1 X(x)Y(y)\Psi_{n,m,n',m'}\ (x,y)dxdy$. The Eq. (3.22) can be expressed as the form

$$u(x,y) = c^T.\Psi(x,y) \tag{3.23}$$

where C and Ψ(x,y) are coefficients matrix and wavelets vector matrix respectively. The number of dimensions of C and Ψ(x,y) are $2^{k-1}2^{k'-1}$MM'x1, and given by

$$\begin{aligned}
C = [&c_{1,0,1,0},\ldots c_{1,0,1,M'-1}, c_{1,0,2,0},\ldots, c_{1,0,2,M'-1},\ldots, c_{1,0,2^{k'-1},0},\ldots,\\
&c_{1,0,2^{k'-1},M'-1},\ldots c_{1,M-1,1,0},\ldots c_{1,M-1,1,M'-1}, c_{1,M-1,2,0},\ldots, c_{1,M-1,2,M'-1},\\
&\ldots, c_{1,M-1,2^{K-1},0},\ldots c_{1,M-1,2^{K-1},M'-1},\ldots c_{2,0,1,0},\ldots c_{2,0,1,M'-1}, c_{2,0,2,0},\\
&\ldots, c_{2,0,2,M'-1},\ldots c_{2,0,2^{K-1},0},\ldots c_{2,0,2^{k-1},M'-1},\ldots c_{2,M-1,1,0}\ldots c_{2,M-1,1,M'-1},\\
&c_{2,M-1,2,0},\ldots, c_{2,M-1,2,M'-1},\ldots, c_{2,M-1,2^{k-1},0},\\
&\ldots, c_{2,M-1,2^{k-1},M'-1},\ldots, c_{2^{k-1},0,1,0},\\
&\ldots c_{2^{k-1},0,1M'-1}, c_{2^{k-1},0,2,0},\ldots, c_{2^{k-1},0,,M'-1},\ldots c_{2^{k-1},0,2^{k-1},0},\\
&\ldots c_{2^{k-1},M-1,2^{k'-1},M'-1}]^T
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\Psi = [&\Psi_{1,0,1,0},\ldots,\Psi_{1,0,1,\mathrm{M}'-1},\Psi_{1,0,2,0},\ldots\Psi_{1,0,2\mathrm{k}-1,0},\ldots\Psi_{1,0,2^{k'-1},\mathrm{M}'-1},\\
&\ldots,\Psi_{1,\mathrm{M}-1,1,0},\ldots\Psi_{1,\mathrm{M}-1,1,\mathrm{M}'-1},\Psi_{1,\mathrm{M}-1,2,0},\ldots,\Psi_{1,\mathrm{M}-1,2,\mathrm{M}'-1},\\
&\ldots\Psi_{1,\mathrm{M}-1,2^{k-1},0},\ldots,\Psi_{1,\mathrm{M}-1,2^{k-1},\mathrm{M}'-1},\ldots,\Psi_{2,0,1,0},\ldots,\\
&\Psi_{2,0,1,\mathrm{M}'-1},\Psi_{2,0,2,0},\ldots\Psi_{2,0,2,\mathrm{M}'-1},\ldots,\Psi_{2,0,2^{k'-1},0},\ldots,\Psi_{2,0,2^{k-1},\mathrm{M}'-1},\ldots,\\
&\Psi_{2,\mathrm{M}-1,1,0},\ldots,\Psi_{2,\mathrm{M}-1,1,\mathrm{M}'-1},\Psi_{2,\mathrm{M}-1,2,0},\ldots,\Psi_{2,\mathrm{M}-1,2,\mathrm{M}'-1},\\
&\ldots,\Psi_{2,\mathrm{M}-1,2^{k'-1},0},\ldots,\\
&\Psi_{2,\mathrm{M}-1,2^{k'-1},\mathrm{M}'-1},\Psi_{2^{k-1},0,1,0},\ldots,\Psi_{2^{k-1},0,1,\mathrm{M}'-1},\Psi_{2^{k-1},0,2,0},\\
&\ldots,\Psi_{2^{k-1},0,2,\mathrm{M}'-1},\ldots,\\
&\Psi_{2^{k-1},0,2^{k-1},0},\ldots\Psi_{2^{k-1},\mathrm{M}-1,2^{k-1},\mathrm{M}'-1}]^T
\end{aligned} \tag{3.25}$$

The integration of the product of two Legendre wavelet function vectors is obtained as

$$\int_0^1\int_0^1 \Psi(x,y)\Psi^T(x,y)dxdy = I \tag{3.26}$$

where I is the identity matrix.
A two-dimensional function f(x,y) defined [0,1) × [0,1) may be expanded by Legendre wavelet series as

$$f(x,y) = \sum_{i=1}^{2^k M} \sum_{j=1}^{2^k M} C_{ij}\ \Psi_i(x)\Psi_j(y) = \Psi^T(x) C \Psi(y) \tag{3.27}$$

where

$$C_{ij} = \int_0^1 f(x,y)\Psi_i(x)dx \int_0^1 f(x,y)\Psi_j(y)dt \tag{3.28}$$

Equation (3.27) can be written into the discrete form (in matrix form) by

$$f(x,y) = \Psi^T(x) C \Psi(y) \tag{3.29}$$

where C and Ψ(t) are $2^{k-1}M \times 1$ matrices given by

$$C = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,2^{k-1}M} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,2^{k-1}M} \\ \vdots & \vdots & \ddots & \vdots \\ c_{2^{k-1}M,0} & c_{2^{k-1}M,1} & \cdots & c_{2^{k-1}M 2^{k-1}M} \end{bmatrix}$$

In the following section, we will give the operational matrix of derivative for two-dimensional Legendre wavelets and give the proof in appendix A.

Theorem 1 *Let Ψ(x,y) be the two-dimensional Legendre wavelets vector defined in Eq. (3.29), we have*

$$\frac{\partial \Psi(x,y)}{\partial x} = D_x \Psi(x,y) \tag{3.30}$$

where D_x *is* $2^{k-1,}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ *and has the form as follows:*

$$D_x = \begin{bmatrix} D & O' & \cdots & 0' \\ 0' & D & \cdots & 0' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \cdots & D \end{bmatrix}$$

In which 0′ and D is $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ *matrix and the element of D is defined as follows:*

$$D_{r,s} = \begin{cases} 2^k \sqrt{(2r-1)(2s-1)} & I, r = 2,3,\ldots M; s = 1,\ldots r-1; r+s \text{ is odd} \\ 0 & otherwise \end{cases} \tag{3.31}$$

and I, O are $2^{k'-1}M' \times 2^{k'-1}M'$ *identity matrix.*

Theorem 2 *Let* $\Psi(x, y)$ *be the two-dimensional Legendre wavelets vector defined in Eq. (3.17), we have*

$$\frac{\partial \Psi(x, y)}{\partial x} = D_y \, \Psi(x,y), \tag{3.32}$$

$$\mathrm{D_y} = \begin{bmatrix} D & O' & \dots & 0' \\ 0' & D & \dots & 0' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \dots & D \end{bmatrix},$$

where $\mathrm{D_y}$ *is* $2^{k-1,}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ *and* O', D *is* $MM' \times MM'$ *matrix is given as*

$$\mathrm{D} = \begin{bmatrix} F & O & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & F \end{bmatrix},$$

in which O and F is $M' \times M'$ *matrix, and F is defined as follows:*

$$F_{r,s} = \begin{cases} 2^{k'}\sqrt{(2r-1)\,(2s-1)}, & r = 2, \dots, M'; S = 1, \dots, r-1; \text{and } r+s \text{ is odd} \\ 0, & otherwise \end{cases} \tag{3.33}$$

By using Eqs. (3.30) and (3.31), the operational matrices for nth derivative can be derived as

$$\frac{\partial^n \Psi(x, y)}{\partial x^n} = D_x^n \Psi(x, y), \frac{\partial^m \Psi(x, y)}{\partial y^m} = D_y^m \Psi(x, y)$$
$$\frac{\partial^{n+m} \Psi(x, y)}{\partial x^n \partial y^m} = D_x^n D_y^m \Psi(x, y)$$

where D^n *is the nth power of matrix D.*

3.6 Block pulse functions (BPFs)

The block pulse functions form a complete set of orthogonal functions which defined on the interval [0, b) by

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m} b \le t < \frac{i}{m} b, \\ 0, & elsewhere \end{cases} \tag{3.34}$$

for i = 1,2,…,m. It is also known that for any absolutely integrable function f(t) on [0,b) can be expanded in block pulse functions:

$$f(t) \cong \xi^T B_m(t) \tag{3.35}$$

$$\xi^T = [f_1, f_2, \ldots, f_m], B_m(t) = [b_1(t), b_2(t), \ldots, b_m(t)] \tag{3.36}$$

where f_i are the coefficients of the block-pulse function, given by

$$f_i = \frac{m}{b} \int_0^b f(t) b_i(t) dt \tag{3.37}$$

Remark 1 Let A and B are two matrices of m × m, then $A \otimes B = (a_{ij} \times b_{ij})_{mm}$.

Lemma 1 *Assuming f(t) and g(t) are two absolutely integrable functions, which can be expanded in block pulse function as f(t) = FB(t) and g(t) = GB(t) respectively, then we have*

$$f(t)g(t) = FB(t)B^T(t)G^T = HB(t) \tag{3.38}$$

where H = F ⊗ G.

3.7 Approximating the nonlinear term

The Legendre wavelets can be expanded into m-set of block-pulse Functions as

$$\Psi(t) = \emptyset_{m\times m} B_m(t) \tag{3.39}$$

Taking the collocation points as following

$$t_i = \frac{i - 1/2}{2^{k-1}M}, i = 1, 2, \ldots, 2^{k-1}M \tag{3.40}$$

The m-square Legendre matrix $\emptyset_{m\times m}$ is defined as

$$\emptyset_{m\times m} \cong [\Psi(t_1)\ \Psi(t_2) \ldots \Psi(t_{2^{k-1}M})] \tag{3.41}$$

The operational matrix of product of Legendre wavelets can be obtained by using the properties of BPFs, let $f(x,t)$ *and* $g(x,t)$ are two absolutely integrable functions, which can be expanded by Legendre wavelets as $f(x,t) = \Psi^T(x)F\Psi(t)$ and $g(x,t) = \Psi^T(x)G\Psi(t)$ respectively.
From Eq. (3.39), we have

$$f(x,t) = \Psi^T(x)F\Psi(t) = B^T(x)\emptyset_{mm}^T F \emptyset_{mm} B(t), \tag{3.42}$$

$$g(x,t) = \Psi^T(x)G\Psi(t) = B^T(x)\emptyset_{mm}^T G \emptyset_{mm} B(t), \tag{3.43}$$

and $F_b = \emptyset_{mm}^T F \emptyset_{mm}$, $G_b = \emptyset_{mm}^T G \emptyset_{mm}$, $H_b = F_b \otimes G_b$.

Then,

$$\begin{aligned} f(x,t)g(x,t) &= B^T H_b B(t), \\ &= B^T(x)\emptyset_{mm}^T inv(\emptyset_{mm}^T) H_b inv(inv(\emptyset_{mm}^T) H_b inv(\emptyset_{mm}))\emptyset_{mm} B(t) \\ &= \Psi^T(x) H \Psi(t) \end{aligned} \tag{3.44}$$

where H = $inv(\emptyset_{mm}^T) H_b inv((\emptyset_{mm}))$

3.8 Function approximation

A given function f(x) with the domain [0,1] can be approximated by:

$$\mathrm{f(x)} = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} c_{k,m} \Psi_{k,m}(x) = C^T . \Psi(x). \tag{3.45}$$

If the infinite series in Eq. (3.45) is truncated, then this equation can be written as:

$$\mathrm{f(x)} \simeq \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} c_{k,m} \Psi_{k,m}(x) = C^T . \Psi(x). \tag{3.46}$$

where C and Ψ are the matrices of size ($2^{\mathrm{j}-1}\mathrm{M} \times 1$).

$$\mathrm{C} = \left[\mathrm{c}_{1,0}, \mathrm{c}_{1,1}, \ldots \mathrm{c}_{1,\mathrm{M}-1}, \mathrm{c}_{2,0}, \mathrm{c}_{2,1}, \ldots \mathrm{c}_{2,\mathrm{M}-1}, \ldots \mathrm{c}_{2,1}^{\mathrm{j}-1}, \ldots \mathrm{c}_{2,\mathrm{M}-1}^{\mathrm{j}-1}\right]^{\mathrm{T}} \tag{3.47}$$

$$\Psi(x) = [\Psi_{1,0}, \Psi_{1,1}, \Psi_{2,0}, \Psi_{2,1}, \ldots \Psi_{2,M-1}, \ldots \Psi_{2^{j-1},M-1}]^T. \tag{3.48}$$

4 Methods of solution

4.1 Solving the NLRDEs by the Haar wavelet method (HWM)

We consider Eq. (2.6)

$$\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - Ks(1 - \alpha s) \tag{4.1}$$

with the initial condition

$$s(x,0) = f(x), 0 \le x \le 1 \tag{4.2}$$

and the boundary conditions

$$s(0,t) = g_0(t), s(1,t) = g_1(t), 0 < t \le T. \tag{4.3}$$

Let us divide the interval (0,1] into N equal parts of length $\Delta t = (0, 1]/N$ and denote $t_s = (s-1)\Delta t, s = 1, 2, \ldots N$. We assume that $\dot{s}''(x,t)$can be expanded in terms of Haar wavelets as formula

$$\dot{s}''(x,t) = \sum_{n=0}^{m-1} c_s(n) h_n(x) = c_{(m)}^T h_{(m)}(x) \tag{4.4}$$

where . and $'$ means differentiation with respect to t and x respectively, the row vector $c_{(m)}^T$ is constant in the subinterval $t \in (t_s, t_{s+1}]$

Integrating formula Eq. (4.4) with respect to t from t_s to t and twice with respect to x from 0 to x, we obtain

$$s''(x,t) = (t-t_s)c_{(m)}^T h_{(m)}(x) + s''(x,t_s) \tag{4.5}$$

$$\begin{aligned} s(x,t) &= (t-t_s)c_{(m)}^T Q_{(m)} h_{(m)}(x) \\ &\quad + s(x,t_s) - s(0,t_s) + x[s'(0,t) - s'(0,t_s)] + s(0,t) \end{aligned} \tag{4.6}$$

$$\dot{s}(x,t) = c_{(m)}^T Q_{(m)} h_{(m)}(x) + x\dot{s}'(0,t) + \dot{s}(0,t) \tag{4.7}$$

By the boundary conditions, we obtain

$$\begin{aligned} s(0,t_s) &= g_0(t_s), \quad s(1,t_s) = g_1(t_s) \\ \dot{s}(0,t) &= g_0'(t), \quad \dot{s}(1,t) = g_1'(t) \end{aligned}$$

Putting $x = 1$ in formulae Eqs. (4.6) and (4.7), we have

$$\begin{aligned} s'(0,t) - s'(0,t_s) &= -(t-t_s)c_{(m)}^T Q_{(m)} h_{(m)}(x) \\ &\quad + g_1(t) - g_0(t) - g_1(t_s) + g_0(t_s) \end{aligned} \tag{4.8}$$

$$\dot{s}'(0,t) = g_1'(t) - c_{(m)}^T Q_{(m)} h_{(m)}(x) - g_0'(t) \tag{4.9}$$

Substituting formulae Eqs. (4.8) and (4.9) into formulae Eqs. (4.4)–(4.6), and discretizising the results by assuming $x \to x_l, t \to t_{s+1}$ we obtain

$$s''(x_l, t_{s+1}) = (t_{s+1} - t_s)c_{(m)}^T h_{(m)}(x_l) + s''(x_l, t_s) \tag{4.10}$$

$$s(x_l, t_{s+1}) = (t_{s+1} - t_s)c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + s(x_l, t_s) - g_0(t_s) + g_0(t_{s+1}) + x_l[-(t_{s+1} - t_s)c_{(m)}^T P_{(m)} f + g_l(t_{s+1}) - g_0(t_{s+1}) - g_1(t_s) + g_0(t_s)] \quad (4.11)$$

$$\dot{s}(x_l, t_{s+1}) = c_{(m)}^T Q_{(m)} h_{(m)}(x) + g_0'(t_{s+1}) + x_l[-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] \quad (4.12)$$

where the vector f is defined as

$$f = [1, \underbrace{0, \ldots, 0}_{(m-1)elements}]^T$$

In the following the scheme

$$\dot{s}(x_l, t_{s+1}) = s''(x_l, t_{s+1}) - K s(x_l, t_{s+1}) \left[1 - \alpha s(x_l, t_{s+1})\right] \quad (4.13)$$

which leads us from the time layer t_s to t_{s+1} is used.

Substituting Eqs. (4.10)–(4.12) into the Eq. (4.13), we obtain

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l[-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1}) = s''(x_l, t_{s+1}) - K s(x_l, t_{s+1})[1 - \alpha s(x_l, t_{s+1})] \quad (4.14)$$

From formula Eq. (4.14) the wavelet coefficients $c_{(m)}^T$ can be successively calculated. Using the relation $s = 1 - p$, we can also obtain the solution of Eq. (2.7).

The exact solution of Eqs. (4.1)–(4.3) by tanh method is given by

$$s(x, t) = \frac{1}{\alpha} - \frac{1}{4\alpha}\left\{1 - \tanh\left[\frac{\sqrt{K/6}}{2}(x - 5\sqrt{K/6}t)\right]\right\}^2 \quad (4.15)$$

It is more convenient to recast the Eq. (4.15) using the relation $1 - \tanh y = 2/[1 + e^{2y}]$ into the following format

$$s(z) = \frac{1}{\alpha}\left\{1 - [1/[1 + \exp(z\sqrt{K/6})]^2]\right\} \quad (4.16)$$

where

$$z = x - 5\sqrt{K/6}t \quad (4.17)$$

Our proposed method (LLWM) can be compared with Rajendran and Senthamarai results (See Ref. [13]).

4.2 Method of solution by the LLWM

We consider the Eq. (4.1)

$$\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - Ks(1-\alpha s) \tag{4.18}$$

Taking Laplace transform on both sides of Eq. (4.18), we get

$$sL(s) - s(x,0) = L[s_{xx} - ks + ks^2] \tag{4.19}$$
$$sL(s) = s(x,0) + L[s_{xx} - ks + ks^2] \tag{4.20}$$
$$L(u) = s^{-1}(s(x,0)) + s^{-1}(L[u_{xx} - ku + ku^2]) \tag{4.21}$$
$$s = s(x,0) + L^{-1}(s^{-1}L[s_{xx} - ks + ks^2]) \tag{4.22}$$

Because

$$L^{-1}[s^{-1}(t^n)] = L^{-1}(n!s^{-(n+2)}) = \frac{1}{n+1}t^{n+1};\ (n = 0, 1, 2, \ldots) \tag{4.23}$$

We have

$$L^{-1}[s^{-1}L()] = \int_0^t (.)dt \tag{4.24}$$

By using the Legendre wavelets method,

$$s = C^T\psi(x,t) \tag{4.25}$$
$$\left.\begin{array}{l} s(x,0) = S^T\psi(x,t) \\ g(s) = G^T\psi(x,t) \end{array}\right\} \tag{4.26}$$
$$C^T\psi(x,t) = S^T\psi(x,t) + C^T Dx^2 - G^T \tag{4.27}$$
$$C^T = S^T + (C^T Dx^2 - G^T)P_t^2 \tag{4.28}$$

Iterative formula is given by

$$s_{n+1} = s(x,0) + \prod (s_{xx} + g(s_x)) \quad where \quad g(s) = ks^2 - ks \tag{4.29}$$

Expanding s(x,t) by Legendre wavelets using the following relation

$$C_{n+1}^T = C_0^T + \left[C_n^T D_x^2 - G_n^T\right] P_t^2 \tag{4.30}$$

5 Convergence analysis and error estimation

$$s^* = s_0 + \prod [s^*_{xx} - g(s^*)] \tag{5.1}$$

and

$$s_{n+1} = s_0 + \prod ((s_n)_{xx} - g(s_n)) \tag{5.2}$$

Subtracting Eq. (5.2)–Eq. (5.1), we obtain

$$s_{n+1} - s^* = \prod [(s_n - s^*)_{xx} - (g(s_n) - g(s^*))] \tag{5.3}$$

Using Lispschitz condition,

$$\left\| g(s_n) - g(s^*) \right\| \leq \gamma \left\| s_n - s^* \right\| \tag{5.4}$$

We have

$$\left\| s_{n+1} - s^* \right\| \leq \left\| \prod (s_n - s^*)_{xx} \right\| - \left\| \prod (g(s_n) - g(s^*) \right\| \tag{5.5}$$

$$\leq \left\| \prod (s_n - s^*)_{xx} - \gamma \right\| - \gamma \left\| s_n - s^* \right\| \tag{5.6}$$

Let

$$s_{n+1} = C^T_{n+1} \psi(x,t) \tag{5.7}$$

$$s^* = C^T \psi(x,t) \tag{5.8}$$

$$\in^T_{n+1} = C^T_{n+1} - C^T \tag{5.9}$$

Equation (5.6) gives

$$\in^T_{n+1} \leq \in^T_n \left\| D^2_x P^2_t + \gamma P^2_t \right\| \tag{5.10}$$

By recursion, we get

$$\in^T_{n+1} \leq \in^T_n \left\| D^2_x P^2_t + \gamma P^2_t \right\|^n \in_0 \tag{5.11}$$

when

$$\lim_{n \to \infty} \left\| D^2_x P^2_t + \gamma P^2_t \right\|^n = 0. \tag{5.12}$$

Table 1 Comparison between the exact and LLWM for Example. 6.1

X	t	U_{exact}	U_{LLWM}
0.25	0.5	0.81839	0.81855
	1.0	0.98292	0.98305
	2.0	0.99988	0.99999
	5.0	1.00000	1.00000
0.50	0.5	0.77590	0.77602
	1.0	0.97815	0.97824
	2.0	0.99985	0.99996
	5.0	1.00000	1.0000
0.75	0.5	0.72582	0.72595
	1.0	0.92207	0.92221
	2.0	0.99981	0.99993
	5.0	1.00000	1.00000

6 Numerical examples

Example 6.1 We consider the Fisher's equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Ku(1-u) \tag{6.1}$$

Subject to the initial condition

$$u(x,0) = \frac{1}{\left(1 + e^{\sqrt{\frac{K}{6}}x}\right)^2} \tag{6.2}$$

Using HAM, the exact solution in a closed form is given by

$$u(x,t) = \frac{1}{\left(1 + e^{\sqrt{\frac{K}{6}}x - \frac{5}{6}at}\right)^2} \tag{6.3}$$

Our proposed method (LLWM) can be compared with Rajendran and Senthamarai results (See Ref. [13]) and Hariharan's results (See Ref. [1]). Comparison between the exact and LLWM for various values of x and t for Example 1. is presented in Table 1. For larger values of k and M, we get the results closer to the real values. Numerical solutions of Eq.(6.1) for various values of x and t are presented in Fig. 1.

Example 6.2 Consider the Fisher's linear equation

$$\mathbf{\partial u/\partial t = \partial^2 u/\partial x^2 - u}, \quad \mathbf{x, t \in R} \tag{6.4}$$

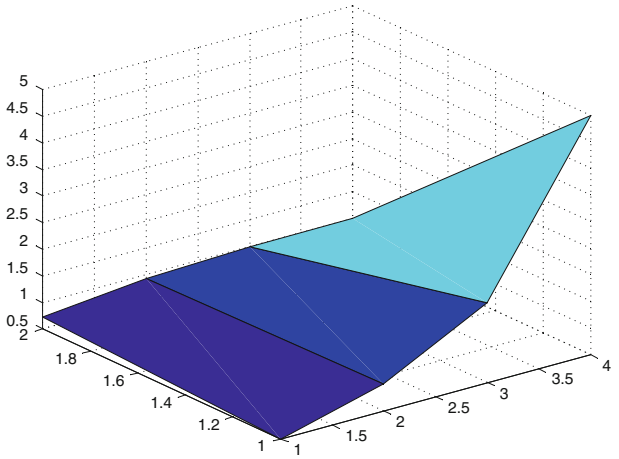

Fig. 1 Numerical solutions of Eq. (6.1) for various values (x, t) with $K = 0.5, k = 1$ *and* $M = 4$

with initial condition

$$\mathbf{u}(\mathbf{x}, \mathbf{0}) = \mathbf{e}^{-\mathbf{x}} + \mathbf{x}, \quad \mathbf{x} \in \mathbf{R} \tag{6.5}$$

Using HAM, the exact solution in a closed form is

$$\begin{aligned} \mathrm{u(x,t)} &= \mathrm{e}^{-\mathrm{x}} + \mathrm{x}((1-\mathrm{t}) + (\mathrm{t}^2/2) - (\mathrm{t}^3/6) + (\mathrm{t}^4/24) - \ldots) \\ &= \mathrm{e}^{-\mathrm{x}} + \mathrm{x}\mathrm{e}^{-\mathrm{t}} \end{aligned} \tag{6.6}$$

The Haar wavelet scheme is given by

$$\begin{aligned} &c^T_{(m)} Q_{(m)} h_{(m)}(x_l) + x_l \left[-c^T_{(m)} P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1}) \right] + g_0'(t_{s+1}) \\ &= u''(x_l, t_{s+1}) - u(x_l, t_{s+1}) \end{aligned} \tag{6.7}$$

Expanding s(x,t) by Legendre wavelets using the following relation

$$C^T_{n+1} = C^T_0 + \left[C^T_n D^2_x - G^T_n \right] P^2_t \tag{6.8}$$

Example 6.3 Consider the following reaction–diffusion equation

$$\partial \mathrm{u}/\partial \mathrm{t} = \partial^2 \mathrm{u}/\partial \mathrm{x}^2 + 2\mathrm{tu}, \quad \mathrm{x,t} \in \mathrm{R} \tag{6.9}$$

with initial condition

$$\mathrm{u(x,0)} = \mathrm{e}^{\mathrm{x}}, \ \ \mathrm{x} \in \mathrm{R} \tag{6.10}$$

Using HAM, the exact solution in a closed form is

$$u(x,t) = e^{x+t+t^2} \tag{6.11}$$

Expanding u(x,t) by Legendre wavelets using the following relation

$$C_{n+1}^T = C_0^T + \left[C_n^T D_x^2 - G_n^T\right] P_t^2 \tag{6.12}$$

Example 6.4 Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - (4x^2 - 2t + 2)u, \quad (x,t) \in \Omega \subset R^2 \tag{6.13}$$

with initial condition

$$u(x,0) = e^{x^2} x \in R \tag{6.14}$$

Using HAM,

$$\begin{aligned}
&\mathrm{u_0(x,t)} = e^{x^2}\\
&\mathrm{u_1(x,t)} = \mathrm{e}^{x^2}(1+\mathrm{t}^2)\\
&\mathrm{u_2(x,t)} = \mathrm{e}^{x^2}(1+\mathrm{t}^2+\mathrm{t}^4/2!)\\
&\mathrm{u_3(x,t)} = \mathrm{e}^{x^2}(1+\mathrm{t}^2+\mathrm{t}^4/2!+\mathrm{t}^6/3!)
\end{aligned}$$

The final solution is

$$\begin{aligned}
\mathrm{u(x,t)} &= \mathrm{e}^{x^2}(1+\mathrm{t}^2+\mathrm{t}^4/2!+\mathrm{t}^6/3!+\mathrm{t}^8/4!+\cdots)\\
&= e^{x^2+t^2}
\end{aligned} \tag{6.15}$$

Expanding s(x,t) by Legendre wavelets using the following relation

$$C_{n+1}^T = C_0^T + \left[C_n^T D_x^2 - G_n^T\right] P_t^2 \tag{6.16}$$

Our proposed method (LLWM) can be compared with Wazwaz and Gorguis results (See Ref. [3])

Example 6.5 Consider the Fisher equation of the form (Wazwaz and Gorguis [3], 2004, Hariharan et al. [1], 2009)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2(1-u), 0 < x < 1 \tag{6.17}$$

With the initial condition

$$u(x,0) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}} \tag{6.18}$$

Using the HAM, the exact solution in a closed form is given by

$$u(x,t) = \frac{1}{1+e^{v(x-vt)}}, v = \frac{1}{\sqrt{2}} \tag{6.19}$$

Our proposed method (LLWM) can be compared with Wazwaz and Gorguis results (See Ref. [3]).

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor x86 Family 6 Model 15 Stepping 13 Genuine Intel ~1596 Mhz.

7 Conclusion

In this work, a new coupled method has been successfully employed to obtain the numerical solutions of linear and NLRDEs arising in mathematical chemistry. The proposed scheme is the capability to overcome the difficulty arising in calculating the integral values while dealing with nonlinear problems. This method shows higher efficiency than the traditional Legendre wavelet method for solving nonlinear PDEs. Numerical example illustrates the powerful of the proposed scheme LLWM. Also this paper illustrates the validity and excellent potential of the LLWM for nonlinear and fractional PDEs. The numerical solutions obtained using the proposed method show that the solutions are in very good coincidence with the exact solution. In addition the calculations involved in LLWM are simple, straight forward and low computation cost.

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